COMPLEX VARIETIES OF GENERAL TYPE WHOSE CANONICAL SYSTEMS ARE COMPOSED WITH PENCILS

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Throughout this paper, most our notations and terminologies are standard within algebraic geometry except for the following which we are in favour of:

$:=$ — definition;

$\sim_{\rm lin}$ — linear equivalence;

$\sim_{\rm num}$ — numerical equivalence.

Let X be a complex nonsingular projective variety of general type with dimension $d (d \geq 2)$. Suppose $\dim \Phi_{|K_X|}(X) = 1$, we usually say that the canonical system $|K_X|$ is composed with a pencil. Taking possible blow-ups $\pi : X' \to X$ according to Hironaka such that $g := \Phi_{|K_X|} \circ \pi$ is a morphism. We have the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, equal] & C \arrow[d, "\psi"] \\ X' \arrow[r, "g"'] \arrow[d, "\pi"'] & W_1 \subset \mathbb{P}^{p_g - 1} \\ X & \end{tikzcd}$$

where we set $W_1 := \overline{\Phi_{|K_X|}(X)}$ and let $g := \psi \circ f$ be the Stein factorization of g. Note that f is a fibration onto a nonsingular curve C. Let F be a general fiber of f, then F is a nonsingular projective variety of dimension $d-1$. We also say that f is a derived fibration of the canonical map. Denote $b := g(C)$, the genus of C.

The aim of this note is to build the following theorems.

Theorem 1. *Let X be a complex nonsingular projective variety of general type with dimension $d (d \geq 3)$. Suppose the canonical system $|K_X|$ be composed with a pencil, using the above diagram and notations, and assume that $b \geq 2$, then either*

$$p_g(F) = 1, \ p_g(X) \geq b - 1,$$

* Supported by NNSFC

or

$$b = p_g(F) = p_g(X) = 2.$$

Remark 1. In the case of dimension 2, one can refer to [10].

Theorem 2. *Under the same assumption as in Theorem 1, assume in addition that* $dim X = 3$ *and* K_X *is nef and big, set* $F_1 := \pi_* F$, *then*

(1) If $b = 1$, *then* $p_g(F) \leq 38$;

(2) If $b = 0$, $K_X \cdot F_1^2 = 0$ *and* $p_g(X) \geq 20$, *then*

$$p_g(F) \leq 38 + \frac{756}{p_g(X) - 19};$$

(3) If $b = 0$ *and* $K_X \cdot F_1^2 > 0$, *then*

$$p_g(F) \geq q(F) \geq \frac{1}{36}(p_g(X) - 1)(p_g(X) - 37).$$

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§1. Proof of Theorem 1

Proof of Theorem 1. Using the first commutative diagram of this paper, we have the fibration $f : X' \to C$. Denote by $\mathcal{L}$ the saturated subbundle of $f_*\omega_{X'}$ which is generated by $H^0(C, f_*\omega_{X'})$, $\mathcal{L}$ is of rank one under the assumption of the theorem. Thus we obtain the following exact sequence

$$0 \to \mathcal{L} \to f_*\omega_{X'} \to \mathcal{Q} \to 0.$$

We know that $R^i f_*\omega_{X'/C}$ is a semi-positive vector bundle(Griffith- Fujita-Kawamata-Ohno semipositivity). $\mathcal{Q} \otimes \omega_C^{-1}$ is automatically semi-positive. Since $H^0(C, \mathcal{L}) = H^0(C, f_*\omega_{X'})$, $H^0(C, \mathcal{Q})$ injects into $H^1(C, \mathcal{L})$. By Riemann-Roch, $h^0(C, \mathcal{Q}) \geq (b-1)(p_g(F)-1)$, $h^1(C, \mathcal{L}) \leq b - 1$, and hence we get $p_g(F) \leq 2$.

If $p_g(F) = 1$, then $p_g(X) = h^0(C, \mathcal{L}) \geq b - 1$ by the semipositivity of $\mathcal{L} \otimes \omega_C^{-1} = f_*\omega_{X'/C}$.

If $p_g(F) = 2$, then we must have $h^0(\mathcal{Q}) = h^1(\mathcal{L}) = b - 1$, so that $\mathcal{Q} \otimes \omega_C^{-1}$ is of degree ≤ 0. Since $f_*\omega_{X'} \otimes \omega_C^{-1}$ is semi-positive, this means that $\mathcal{L}$ is of degree $\geq 2b - 2$, with non-trivial H^1. Hence $\mathcal{L} \cong \omega_C$, and $1 = h^1(C, \mathcal{L}) \geq h^0(\mathcal{Q}) = b - 1$. This shows that $b = 2$, $p_g(X) = h^0(C, \mathcal{L}) = 2$, completing the proof of Theorem 1.

§2. Proof of Theorem 2

At first, let us recall Miyaoka's inequality as follows.

Fact. *([7]) Let* X *be a nonsingular projective threefold with nef and big canonical divisor* K_X. *Then* $3c_2 - c_1^2$ *is pseudo-effective, where* c_i*'s are Chern numbers of* X.

Proposition 2.1. *Let* X *be a nonsingular projective 3-fold with nef and big canonical divisor* K_X. *Assume that* $|K_X|$ *be composed with a pencil and that* $K_X \cdot F_1^2 = 0$, *then*

$$\mathcal{O}_F(\pi^*(K_X)|_F) \cong \mathcal{O}_F(\sigma^*(K_{F_0})),$$

where $\sigma : F \to F_0$ is the contraction onto the minimal model.

Proof. This can be obtained by a similar argument as that of Theorem 7 in [6].

PROOF OF THEOREM 2. We use the same exact sequence as in the proof of Theorem 1. It is easy to see that $K_{X'} \sim_{\mathrm{num}} (\deg \mathcal{L})F + Z$, where Z is the fixed part.

Case (1). $b = 1$. In this case, we can suppose $X' = X$. $\mathcal{Q}$ is semi-positive. Note that $\deg \mathcal{Q} = 0$; otherwise, by Riemann-Roch, we should have

$$h^0(C, f_*\omega_X) \geq \deg \mathcal{L} + \deg \mathcal{Q} > \deg \mathcal{L} = h^0(C, \mathcal{L}),$$

a contradiction. $R^1 f_*\omega_X$ is semi-positive, while $R^2 f_*\omega_X \cong \mathcal{O}_C$ by duality. Thus

$$\begin{aligned}\chi(X, \omega_X) &= \chi(C, f_*\omega_X) - \chi(C, R^1 f_*\omega_X) + \chi(C, R^2 f_*\omega_X)\\ &\leq \chi(C, f_*\omega_X) = \deg \mathcal{L} = p_g(X).\end{aligned}$$

It follows from Miyaoka's inequality that $K_X^3 \leq 72 \deg \mathcal{L}$. On the other hand,

$$\begin{aligned}K_X^3 &\geq (\deg \mathcal{L}) K_X^2 \cdot F\\ &= (\deg \mathcal{L}) K_F^2 \geq 2(\deg \mathcal{L})(p_g(F) - 2).\end{aligned}$$

This implies that $(\deg \mathcal{L})(p_g(F) - 2) \leq 36 \deg \mathcal{L}$, i.e. $p_g(F) \leq 38$.

Case (2). $b = 0$ and $K_X \cdot F_1^2 = 0$. In this case, any vector bundle on C is a direct sum of line bundles. $f_*\omega_{X'}$ is a direct sum of $\mathcal{L}$ and $p_g(F) - 1$ line bundles of degree -1 or -2, while $R^1 f_*\omega_{X'}$ is a direct sum of $q(F)$ line bundles of degree ≥ -2. Hence

$$\begin{aligned}\chi(X', \omega_{X'}) &= \chi(C, f_*\omega_{X'}) - \chi(C, R^2 f_*\omega_{X'}) + \chi(C, \omega_C)\\ &\leq \deg \mathcal{L} + 1 + q(F) - 1.\end{aligned}$$

Then, by Proposition 2.1, we get

$$(\deg \mathcal{L})(p_g(F) - 2) \leq \frac{1}{2}(\deg \mathcal{L}) K_{F_0}^2 \leq 36(\deg \mathcal{L} + q(F)).$$

Apply the inequality $p_g(F) \geq 2q(F) - 4$ ([2]), we get

$$(\deg \mathcal{L} - 18) p_g(F) \leq 38 \deg \mathcal{L} - 72.$$

We obtained the desired inequality by substituting $\deg \mathcal{L}$ by $p_g(X) - 1$.

Case (3). $b = 0$ and $K_X \cdot F_1^2 > 0$. It is easy to check that $K_X \cdot F_1^2$ is an even number. We get the following inequality

$$K_X^3 \geq (\deg \mathcal{L})^2 K_X \cdot F_1^2 \geq 2(\deg \mathcal{L})^2.$$

Therefore we have

$$2(\deg \mathcal{L})^2 \leq 72(\deg \mathcal{L} + q(F)),$$

which directly induces what we want. The proof is completed.

§3. Examples

3.1 Examples with $p_g(F) = 1$ and $p_g(X) \geq b$.

Let S be a minimal surface with $K_S^2 = 1$ and $p_g(S) = q(S) = 0$. We chose S in such a way that S admits a torsion element η of order 2, i.e., $\eta \in Pic^0 S$ and $2\eta \sim_{\text{lin}} 0$. For the existence of this surface, we may refer to [1] or [9]. Let C be a nonsingular curve of genus $b \geq 1$. Set $Y := C \times S$. Let $p_1 : Y \longrightarrow C$ and $p_2 : Y \longrightarrow S$ be the two projection maps. Let D be an effective divisor on C with $\deg D = a > 0$. Let $\delta = p_1^*(D) + p_2^*(\eta)$, $B \sim_{\text{lin}} 2\delta \sim_{\text{lin}} p_1^*(2D)$, we can take B composed of $2a$ distinct points. The pair (δ, B) determines a smooth double cover X over Y. We have the following commutative diagram, where π is the double covering.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, equal] & Y = C \times S \arrow[r, "p_2"] \arrow[d, "p_1"] & S \\ X \arrow[r, "\Psi"'] & C & \end{tikzcd}$$

We can see that $\Phi_{|K_X|}$ factors through Ψ. These examples satisfy $p_g(X) = a + b - 1$, $K_X^3 = 6(a + 2b - 2)$, $h^2(\mathcal{O}_X) = 0$, $q(X) = b$, $K_F^2 = 2$, $p_g(F) = 1$ and $q(F) = 0$.

3.2 Examples with $p_g(F) = 1$ and $p_g(X) = b - 1$.

In the construction of 3.1, take C be a hyperelliptic curve of genus $b \geq 3$. Let $\tau = p - q$ be a divisor on C such that $2\tau \sim_{\text{lin}} 0$. Take $\delta = p_1^*(\tau) + p_2^*(\eta)$, then $2\delta \sim_{\text{lin}} 0$. Therefore δ determines an unramified double cover $\pi : X \longrightarrow Y$. X is an example with $p_g(X) = b - 1$, $h^2(\mathcal{O}_X) = 0$, $q(X) = b$ and $K_X^3 = 6(2b - 2)$.

3.3 Examples with $b = 0$ and $p_g(F) = 2$.

In the construction of 3.1, take S be a minimal surface S with $K_S^2 = 2$ and $p_g(S) = q(S) = 1$ and take $C = \mathbb{P}^1$. Take an effective divisor D on C with $\deg D = a \geq 3$, $\eta \in Pic^0 S$ with $2\eta \sim_{\text{lin}} 0$. Denote by $\delta := p_1^*(D) + p_2^*(\eta)$ and $R \sim_{\text{lin}} 2\delta \sim_{\text{lin}} p_1^*(2D)$. Thus the pair (δ, R) determines a double covering $\pi : X \to Y$. We can check that X is an example with $p_g(X) = a - 1$, $K_X^3 = 6(a - 2)$, $b = 0$, $K_F^2 = 4$, $q(F) = 1$ and $p_g(F) = 2$.

3.3 Examples with $b = 1$ and $p_g(F) = 2$.

Let S be a minimal surface with $p_g(S) = q(S) = 1$ and $K_S^2 = 2$. We know that the albanese map of S is just a genus two fibration onto an elliptic curve. It can be constructed from a double cover onto a ruled surface P which is over the elliptic curve with invariant $e = -1$. Furthermore, all the singularities on the branch locus corresponding to this double cover are negligible. Let $\pi_0 : S \to \tilde{P}$ be this double cover with covering data (δ_0, R_0). Note that $q(P) = 1$.

Take an elliptic curve E and denote $T := E \times \tilde{P}$. Let p_1 and p_2 be two projection maps. Take a 2-torsion element $\eta \in \mathrm{Pic}^0 E$. Since $\tilde{P}$ is a fibration over an elliptic curve, we can take a 2-torsion element $\tau \in \mathrm{Pic}^0 \tilde{P}$ such that $\pi_0^*(\tau) \not\sim_{\text{lin}} 0$ through the double cover π_0.

Let $\delta_1 = p_1^*(\eta) + p_2^*(\delta_0)$ and $R_1 = 2\delta_1$, then the pair (δ_1, R_1) determines a double cover $\Pi_1 : Y \to T$. Let $\phi = p_1 \circ \Pi_1$. Take a divisor A on E with $\deg A = a > 0$. Let $\delta_2 = \phi^*(A) + \Pi_1^* p_2^*(\tau)$ and $R_2 = 2\delta_2$, then the pair (δ_2, R_2) determines a smooth double cover $\Pi_2 : X \to Y$. We can see that X is a minimal threefold of general type and the

canonical system $|K_X|$ is composed with a pencil. $\Phi_{|K_X|}$ factors through Π_2 and ϕ. This example satisfies $b = 1$, $p_g(F) = 2$, $q(F) = 1$, $K_F^2 = 16$, $p_g(X) = a$, $q(X) = 2$ and $K_X^3 = 12a$.

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